

Electromagnetic Monitoring of Internal Structure of Three-Dimensional Objects

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Abstract - This paper deals with some multidimensional inverse dynamic problems for electrodynamic equations. These problems are reduced to standard tomography problems. As a result of these reductions there are uniqueness, stability estimate of the solution and efficient numerical methods.

1. INTRODUCTION

As an emergent technology, industrial process tomography is now about 10 years old. At first X-ray tomography found its application in medicine and then in industry. At present, tomography is intensively employed in non-destructive testing and flaw detection. The need for a simple, cheap, and environmental-friendly technology of monitoring of internal structure of objects gave start to develop new methods of tomography. There is now growing interest to such an area as electromagnetic tomography. There are basic principles of 3-dimensional electromagnetic monitoring by measuring the dynamic of electromagnetic field on an enclosing 3-dimensional object. These principles are based on the following recent research results:

- 1) recovering structure of inhomogeneous electromagnetic media by the measurement of the dynamic of electromagnetic fields in domains accessible to observation;
- 2) constructing efficient and stable algorithms solving different problems of computerized tomography;
- 3) a possibility to measure the dynamic of electromagnetic fields with high accuracy and resolution.

Below we will consider in detail the first of above-mentioned principles on the example of the problem of electrical conductivity determination of inhomogeneous electromagnetic medium which is inside a 3-dimensional ball, with given dynamic of the electromagnetic field on the surface of this ball. We show that this problem can be reduced to a computerized tomography for which we can use well-known, efficient, stable numerical methods (see, for instance [1]-[2]).

The present paper deals with the inverse problem of determining the conductivity function $\mathbf{s}(x)$ which enters, as a coefficient, into the Maxwell system with constant dielectric and magnetic permeabilities. We describe here the formulation of the direct and inverse problems for

this Maxwell system and properties of the direct problem solution. Using these properties we show that the inverse problem can be reduced to the well-known tomography problem.

2. DIRECT PROBLEM AND PROPERTIES OF ITS SOLUTION

Let us consider for $x=(x_1, x_2, x_3) \in R^3, t \in R$ the following Maxwell system

$$\begin{aligned} \operatorname{curl} H &= \mathbf{e} \left(\frac{\partial E}{\partial t} + \mathbf{s}(x) E \right) + j, \\ \operatorname{curl} E &= -\mathbf{m} \frac{\partial H}{\partial t} \end{aligned} \tag{1}$$

and the conditions

$$E|_{t<0} = 0, \quad H|_{t<0} = 0. \tag{2}$$

Here $E=(E_1, E_2, E_3), H=(H_1, H_2, H_3)$ are the vector-functions of electric and magnetic intensity.

Our assumptions are the following. The quantities $\mathbf{e}, \mathbf{m}, l$ are fixed positive constants, $\mathbf{s}(x)$ is a smooth function from the class $C^{\mathbf{y}}(R^3)$ with constant values $\mathbf{s} \equiv \mathbf{s}^0$ for $x \in (R^3 \setminus D)$, where D is open in R^3 , and $D \subset D(l) = \{x \in R^3 \mid |x| < l\}$;

$$S_l = \{x \in R^3 \mid |x| = l\}, \quad D_1 = R^3 \setminus D(l)$$

$$\mathbf{t}(x, x^0) = \mathbf{e}\mathbf{m}|x - x^0|, \quad \Gamma = t^2 - \mathbf{t}^2(x, x^0)$$

$\mathbf{q}_0(t)$ is the Heaviside step function,

$$\mathbf{q}_k(t) = \frac{t^k}{k!} \mathbf{q}_0(t), \quad \mathbf{q}_{-k}(t) = \frac{d^k}{dt^k} \mathbf{q}_0(t), \quad k = 1, 2, \dots$$

$\mathbf{d}(x)$ is the Dirac delta-function,

$$\mathbf{d}(x - x^0) = \mathbf{d}(x_1 - x_1^0) \cdot \mathbf{d}(x_2 - x_2^0) \cdot \mathbf{d}(x_3 - x_3^0),$$

$x^0 = (x_1^0, x_2^0, x_3^0)$ is a parameter from R^3 .

Theorem 1. Let $j = \mathbf{e}\mathbf{d}(x - x^0)\mathbf{d}(t), x^0 \in D_1, \mathbf{e}$ be a vector in R^3 . Then under the above

conditions and notations there exists the unique solution of the problem (1), (2) which can be represented in the form

$$H(x, t, x^0) = \begin{cases} \sum_{k=2}^{\infty} a^k(x, t, x^0) \mathbf{q}_k(t^2 - t^2(x, x^0)), & t > t_1 \\ \tilde{h}(x, t, x^0), & t \leq t_1 \end{cases} \quad (3)$$

$$E(x, t, x^0) = \begin{cases} \sum_{k=2}^{\infty} b^k(x, t, x^0) \mathbf{q}_k(t^2 - t^2(x, x^0)), & t > t_1 \\ \tilde{\ell}(x, t, x^0), & t \leq t_1 \end{cases} \quad (4)$$

where

$$t_1 = \frac{\text{dist}(D, S(l))}{2\sqrt{\mathbf{em}}}$$

$$\tilde{h}(x, t, x^0) = \frac{1}{2\mathbf{p}} (\mathbf{em})^{1/2} \exp\left(-\frac{\mathbf{s}_0 t}{2}\right) \mathbf{q}_0(t) \times \text{curl} \left[\left(\mathbf{d}(\Gamma) + \mathbf{q}_0(\Gamma) \exp\left(\frac{\Gamma(\mathbf{s}_0/4)^2 - 1}{\Gamma}\right) \right) \mathbf{e} \right];$$

$$\begin{aligned} \tilde{\ell}(x, t, x^0) &= \frac{1}{2\mathbf{p}} (\mathbf{em})^{1/2} \exp\left(-\frac{\mathbf{s}_0 t}{2}\right) \mathbf{q}_0(t) \times \\ &\times \left[\left(\frac{2t}{\Gamma} - 2t + \frac{\mathbf{s}_0}{2} \right) \mathbf{d}(\Gamma) + \left(-\frac{2t}{\Gamma^2} + \frac{\mathbf{s}_0}{2\Gamma} \right) \times \right. \\ &\times \exp(\Gamma(\mathbf{s}_0/4)^2) \mathbf{q}_0(\Gamma) + \left. \left(\frac{2t(\mathbf{s}_0/4)^2}{\Gamma} + \frac{2t}{\Gamma^2} - \frac{\mathbf{s}_0}{2\Gamma} \right) \mathbf{q}_0(\Gamma) \right] \mathbf{e}; \end{aligned}$$

$a^k, b^k \in C^{\mathbf{Y}}(R^3 \times [t_1, T] \times \overline{D_1})$ for every $T > 0$.

Moreover, the following formulas for the functions $b^{-2}(x, t, x^0), a^{-2}(x, t, x^0)$ take place

$$b^{-2}(x, t, x^0) = -2t \mathbf{a}^{-1}(x, t, x^0) - \mathbf{b}^{-2}(x, t, x^0) \quad (5)$$

$$a^{-2}(x, t, x^0) = \frac{1}{\mathbf{m}} [\mathbf{a}^{-1}(x, t, x^0) \times \tilde{\mathbf{N}} t^2(x, x^0)] \quad (6)$$

where

$$\mathbf{a}^{-1}(x, t, x^0) = \frac{1}{2\mathbf{p}} (\mathbf{em}^3)^{1/2} \times \exp\left(-\frac{t}{2|x-x^0|} \int_{L(x, x^0)} \mathbf{s}(\mathbf{x}) ds\right) \mathbf{e}; \quad (7)$$

$$\begin{aligned} \mathbf{b}^{-2}(x, t, x^0) &= -\frac{2}{t} t^2(x, x^0) \times \\ &\times (\mathbf{a}^{-1}(x, t, x^0) \cdot \mathbf{n}(x, x^0)) \mathbf{n}(x, x^0); \end{aligned} \quad (8)$$

here

$$t^2(x, x^0) = \mathbf{em} |x - x^0|^2, \quad \mathbf{n} = (x - x^0) / |x - x^0|.$$

Remark 1. Theorem 1 can be proved using the reasoning from works [3, 4, 5].

Let $(E^k(x, t, x^0), H^k(x, t, x^0))$ be the solution of the problem (1)-(2) for $e = e_k$ where e_k is a basis vector of R^3 ; $f(x, x^0)$ and $g(x, x^0)$ be the functions defined as

$$g(x, x^0) = \lim_{t \rightarrow t(x, x^0) + 0} \sum_{k=1}^2 \int_0^t E^k(x, z, x^0) \times (z^2 - t^2(x, x^0)) dz^2;$$

$$f(x, x^0) = \lim_{t \rightarrow t(x, x^0) + 0} \sum_{k=1}^2 \int_0^t H^k(x, z, x^0) \times (z^2 - t^2(x, x^0)) dz^2.$$

Using formula (3)-(7) and the fact that $\mathbf{s}(x)$ is identical constant for $x \in \hat{I} R^3 \setminus D$ we obtained the following properties:

$$g(x, x^0) = \frac{\mathbf{m}}{\mathbf{e}} \Psi(x, x^0) \sum_{k=1}^2 |e_k - (e_k \cdot \mathbf{n}(x, x^0)) \mathbf{n}(x, x^0)|^2; \quad (9)$$

$$f(x, x^0) = \Psi(x, x^0) \sum_{k=1}^2 |e_k \times \mathbf{n}(x, x^0)|^2 \quad (10)$$

where

$$\Psi(x, x^0) = \frac{(\mathbf{em})^2}{4\mathbf{p}^2} \exp\left(-\sqrt{\mathbf{em}} \int_{L(x, x^0)} \mathbf{s}(\mathbf{x}) ds\right) \quad (11)$$

3. INVERSE PROBLEM IN RAY FORMULATION AND ITS REDUCTION TO A TOMOGRAPHY PROBLEM

Let us consider positive numbers $l, T, \mathbf{e}, \mathbf{m}$ such that $T > 2l\mathbf{em}$. We will suppose that the function $\mathbf{s}(x) \in C^{\mathbf{Y}}(R^3)$ has constant values \mathbf{s}^0 for $x \in \hat{I} R^3 \setminus D$ and is unknown for $x \in \hat{I} D$. We will also use here the notation of the previous section. The main object of this section is the following inverse problem.

Inverse Problem. Find the unknown function $\mathbf{s}(x)$ for $x \in \hat{I} D$ which enters into the system of differential equations (1), (2) if we know one of the following additional information

$$E^k(x, x^0, t)|_{x \in S_l, x^0 \in S_l} = g^k(x, x^0, t), k=1,2 \quad (12)$$

or

$$H^k(x, x^0, t)|_{x \in S_l, x^0 \in S_l} = f^k(x, x^0, t), k=1,2 \quad (13)$$

Here $g^k(x, x^0, t)$, $f^k(x, x^0, t)$, $k=1, 2$, are unknown functions, for $x \in S_l$, $x^0 \in S_l$, $t \in [0, T]$.

We note that for

$$\mathbf{n}(x, x^0) = (x - x^0) / |x - x^0|, \quad x \in S_l, \quad x^0 \in S_l$$

the following inequalities take place

$$\sum_{k=1}^2 |e_k - (e_k \cdot \mathbf{n}(x, x^0))\mathbf{n}(x, x^0)|^2 > 0$$

$$\sum_{k=1}^2 |e_k \times \mathbf{n}(x, x^0)|^2 > 0$$

and therefore (9)-(11) yield the relations:

$$\int_{L(x, x^0)} \tilde{\mathbf{s}}(\mathbf{x}) d\mathbf{s} = F_1(x, x^0), \quad x \in S_l, \quad x^0 \in S_l \quad (14)$$

$$\int_{L(x, x^0)} \tilde{\mathbf{s}}(\mathbf{x}) d\mathbf{s} = F_2(x, x^0), \quad x \in S_l, \quad x^0 \in S_l \quad (15)$$

where

$$\tilde{\mathbf{s}}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) - \mathbf{s}_0,$$

$$F_1(x, x^0) = \frac{1}{\sqrt{\mathbf{em}}} \ln \left(\frac{(\mathbf{em})^2}{4p^2 f(x, x^0)} \sum_{k=1}^2 |e_k \times \mathbf{n}(x, x^0)|^2 \right) - \int_{L(x, x^0)} \mathbf{s}^0 d\mathbf{s};$$

$$F_2(x, x^0) = \frac{1}{\sqrt{\mathbf{em}}} \ln \left(\frac{(\mathbf{em})^2}{4p^2 g(x, x^0)} \sum_{k=1}^2 |e_k - (e_k \cdot \mathbf{n}(x, x^0))\mathbf{n}(x, x^0)|^2 \right) - \int_{L(x, x^0)} \mathbf{s}^0 d\mathbf{s}.$$

We can determine the values of the function $F_l(x, x^0)$ for every $x \in S_l$, $x^0 \in S_l$ from the information of the form (13). Thus the inverse problem in this case can be reduced to the tomography problem (14). Analogously, the inverse problem with the additional information of the form (12) can be reduced to the tomography problem (15).

4. CONCLUSION

Sections 2 and 3 contain the mathematical statement and apparatus of the electrical conductivity determination problem for nonhomogeneous electromagnetic medium. We suppose that dielectric and magnetic permeabilities of this medium \mathbf{m} , \mathbf{e} are constants, and the conductivity $\mathbf{s}(x)$ is the smooth function of a space point x . Moreover, we assume also, that this function $\mathbf{s}(x)$ equals to \mathbf{s}^0 (a constant) outside some domain D , and is unknown function

of three arguments inside D . The boundary of the domain D is unknown also, but it is known, that this domain D is included in a ball of some fixed radius l , i.e.

$$D \subset \{x \in R^3 / |x| < l\}.$$

The current density has the form

$$\mathbf{J} = \frac{\mathbf{s}(x)}{\mathbf{e}} \mathbf{E} + \mathbf{J}^0,$$

here \mathbf{J}^0 is the external current density having the form

$$\mathbf{J}^0 = e \mathbf{d}(x - x^0) \times \mathbf{d}(t),$$

where e is a fixed vector of the 3-dimensional space; x^0 is a point of the sphere $S_l = \{x \in R^3 / |x| = l\}$.

Notice that this point x^0 can run the surface S_l being the parameter of the experiment;

$$\mathbf{d}(x - x^0) = \mathbf{d}(x_1 - x_1^0) \cdot \mathbf{d}(x_2 - x_2^0) \cdot \mathbf{d}(x_3 - x_3^0)$$

where $\mathbf{d}(x)$ is the Dirac-delta function.

Remark 2. The internal current of the form $\mathbf{J}^0 = e \mathbf{d}(x - x^0) \times \mathbf{d}(t)$ is a pulse directional source which is concentrated in the point $x = x^0$ at the moment $t = 0$ in the direction of the vector e .

We will denote through $E^k(x, t, x^0)$, $H^k(x, t, x^0)$ the electric and magnetic intensity vectors corresponding to the external current \mathbf{J}^0 with the directional vector $e = e^k$, where e^k is k -th basis vector from R^3 , i.e. $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, $e^3 = (0, 0, 1)$.

We determine the function $F(x, x^0, t)$, $G(x, x^0, t)$ by equalities

$$G(x, x^0, t) = \sum_{k=1}^2 \left| \int_0^t E^k(x, z, x^0) (z^2 - t^2(x, x^0)) dz \right|^2,$$

$$F(x, x^0, t) = \sum_{k=1}^2 \left| \int_0^t H^k(x, z, x^0) (z^2 - t^2(x, x^0)) dz \right|^2,$$

where

$$t^2(x, x^0) = \mathbf{em} \sum_{i=1}^2 (x_i - x_i^0)^2.$$

The main objective of the study is to determine the domain D and the function $\mathbf{s}(x)$ for

$x \hat{\Gamma} D$ if it is known one of the functions $F(x, x^0, t)$ or $G(x, x^0, t)$ for all $x \hat{\Gamma} S_b, x^0 \hat{\Gamma} S_b$,

$$t \hat{\Gamma} [t(x, x^0) - w, t(x, x^0) + w],$$

where w is a fixed positive number.

Let $g(x, x^0), f(x, x^0), n(x, x^0)$ be the functions defined by the equalities

$$g(x, x^0) = \lim_{t \rightarrow t(x, x^0)_+0} G(x, x^0, t),$$

$$f(x, x^0) = \lim_{t \rightarrow t(x, x^0)_+0} F(x, x^0, t),$$

$$n(x, x^0) = \frac{x - x^0}{|x - x^0|}.$$

It was shown in sections 2 and 3 that problem can be reduced to standard tomography problem (14) or (15). We remark here that the right-hand sides of the equalities (14), (15) are defined though the functions $f(x, x^0), g(x, x^0)$, respectively. To solve the obtained tomography problems (14) and (15), it can be used recent numerical techniques such as [1]-[2]. A possibility to measure the dynamic of electromagnetic fields is the separate important topic which we do not discuss in our present paper having the theoretical character.

We conclude by remarking that, for different applications, the original statement of the problem may be quite different. It is possible to reduce a lot of problems of that kind to the standard computerized tomography problem.

The presented technology of electromagnetic monitoring can be applied in medicine, non-destructive testing, flaw detection.

From mathematical point the results of the present paper continue the investigation of the work [3-6].

REFERENCES

- [1] Natterer F. The Mathematics of Computerized Tomography -Stuttgart, John Wiley & Sons Ltd and B.G. Teubner, 1986.-222 p.
- [2] Trofimov O. Numerical Algorithms of Tomography on Base Distributions // Proceedings of 15-th World Congress on Scientific Computation, Modelling and Applied Mathematics Berlin, August 1997, Wissenschaft & Technik Verlag-v.3,-p.707-711.
- [3] Romanov V.G. Structure of the Fundamental Solution of the Cauchy Problem for the Maxwell System of Equations // Differents. Uravn. ,1986 vol.22, N.9, 1577-1587 (in Russian).
- [4] Romanov V.G. Inverse Problems of Mathematical Physics. Nauka, Moscow, 1984 (in Russian).
- [5] Yakhno V.G.. Inverse Problem for Differential Equations of Elasticity. Novosibirsk, Nauka (1990). (In Russian)
- [6] Yakhno V.G. Multidimensional Inverse Problems in Ray Formulation for Hyperbolic Equations // Inverse and Ill-posed Problems, VSP, Utrecht, The Netherlands,1998-v.6, p. 71-85.